**MIXED RECTANGULAR FINITE ELEMENTS FOR PLATE BENDING**

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The paper describes three different rectangular plate bending elements based on Reissner's type stationary variational principles. They differ in the number of dependent variables approximated independently and also in the number of nodes per element. The first element type treats the transverse deflection and the three moments as unknowns at each of the corner nodes; the second element type treats the transverse deflection and two normal moments as unknowns at the corner nodes; the third one treats the transverse deflection as unknown at the corner node, and the moments \( m_x \) and \( m_y \) at the midnodes of opposite sides of the rectangle. These three types of elements are used to solve square plate problems with various boundary conditions and loadings.

**INTRODUCTION**

Owing to the severe continuity requirements placed on the trial functions employed in the conventional (conforming) finite element models of plate bending (derived with the total potential energy principle), the resulting element matrices are algebraically complex and consequently their solutions require large amounts of computing time. Further, the moments (or stresses) computed using the conventional plate bending elements are not accurate. To avoid these problems Herrmann (1) suggested a new triangular plate bending element which treats the transverse deflection and normal and tangential moments as unknown dependent variables. This element was derived using Reissner's variational principle for a thin plate bending element. Reissner's variational principle yields, as Euler equations, the moment-equilibrium equations and the kinematic relations connecting the transverse displacement (and its derivatives) to the moments. These equations are lower order (2nd order) compared to the fourth-order (biharmonic) equation governing the transverse displacement of the plate. This attractive feature relaxes the continuity requirements on the trial functions. Several mixed finite element models have been derived based on variants of Reissner's functional and/or using various order polynomial approximations (see, for example, (2-6) ). The present paper describes construction and applications of three mixed rectangular finite elements.

**FORMULATION**

Let the middle plane of the plate to be analyzed be denoted \( \Omega \subset \mathbb{R}^2 \) with \( \Omega \) piecewise smooth boundary \( \partial \Omega \). This region is decomposed into finite elements \( \{ \Omega_e \}^N_{e=1} \). Let \( w \) denote the transverse deflection, \( m_x \), \( m_y \), and \( m_{xy} \) the moments, and \( P \) the lateral load in the plate. The kinematic relations are given by

\[
\begin{align*}
\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} &= 0 \quad \text{Eq. 1} \\
\frac{\partial^2 m_x}{\partial x \partial y} &= 0 \\
\frac{\partial^2 m_y}{\partial x \partial y} &= 0 \\
\frac{\partial^2 m_{xy}}{\partial x \partial y} &= 0 \\
\end{align*}
\]

where \( S = 12/Et^3 \), \( E \) being the Young's modulus, \( t \) the thickness, and \( \mu \) is the Poisson's ratio of the plate. The equilibrium equation is given by

\[
\begin{align*}
\frac{\partial^2 m_x}{\partial x^2} + 2\frac{\partial^2 m_{xy}}{\partial x \partial y} + \frac{\partial^2 m_y}{\partial y^2} &= P \quad \text{Eq. 2} \\
\end{align*}
\]

These equations must be adjoined by appropriate conditions on the boundary of the plate. We introduce the notation

\[
\begin{align*}
\sigma_{xx} &= \frac{\partial^2 w}{\partial x^2} \\
\sigma_{yy} &= \frac{\partial^2 w}{\partial y^2} \\
\sigma_{xy} &= \frac{\partial^2 m_{xy}}{\partial x \partial y} \\
\end{align*}
\]

where \( n_x = \cos(n,x) \) and \( n_y = \cos(n,y) \) are the direction cosines of the outward normal \( n = (n_x, n_y) \) on \( \partial \Omega \). We specify the following set of boundary conditions.

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(a) essential boundary conditions:
\[ \psi = \hat{\psi} \text{ on } \partial \Omega \text{, } m = \hat{m} \text{ on } \partial \Omega \]  
Eq. 4

(b) natural boundary conditions:
\[ q = \hat{q} \text{ on } \partial \Omega, \quad \frac{\partial \psi}{\partial n} = \hat{m} \text{ on } \partial \Omega \]  
Eq. 5

Here variables with "^" denote specified values, and \( \partial \Omega_w \) and \( \partial \Omega_m \) are disjoint sets whose union is \( \partial \Omega \).

A variational formulation of Equations 1-5 has been derived (7) and is given by

\[ R_1^e (\psi, m, x, y, m_{xy}) = \sum_{e=1}^{N} R_1^e (\psi^e, m^e, x^e, y^e, m_{xy}^e) \]

Here \( R_1^e \) is the restriction of the functional \( R_1 \) to element \( e \), and

\[ R_1^e (\psi^e, m^e, x^e, y^e, m_{xy}^e) = \int_{\Omega_e} \left[ -\frac{1}{2} \left( \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} - 2 \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial y} + 2(1 + v) \frac{\partial^2 \psi}{\partial x^2} \right) + \frac{3}{2} \frac{\partial^2 m}{\partial x^2} + \frac{3}{2} \frac{\partial^2 m}{\partial y^2} + \frac{3}{2} \frac{\partial^2 m}{\partial x \partial y} \right] dx dy \]

wherein for the sake of brevity the element label 'e' is omitted. This functional can be used to construct independent approximations of \( w, m_x, m_y \) and \( m_{xy} \). If we assume that the third equation in Equation 1 is identically satisfied (i.e., eliminating \( m_{xy} \)), we obtain from \( R_1^e \),

\[ R_2^e (\psi^e, m^e, x^e, y^e) = \int_{\Omega_e} \left[ -\frac{1}{2} \left( \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} - 2 \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial y} + 2(1 + v) \frac{\partial^2 \psi}{\partial x^2} \right) + \frac{3}{2} \frac{\partial^2 m}{\partial x^2} + \frac{3}{2} \frac{\partial^2 m}{\partial y^2} + \frac{3}{2} \frac{\partial^2 m}{\partial x \partial y} \right] dx dy \]

wherein the boundary terms are omitted temporarily. Functional in Equation 7 can be used to construct independent approximations of \( w \), \( m_x \) and \( m_y \).

**Mixed Model I.**

Functional \( R_1 \) is employed to construct the rectangular finite element. Bilinear approximations are used for each variable. Thus the element has four nodes and four degrees of freedom at each node (see Figure 1a), resulting in a 16 by 16 element stiffness matrix. The output contains \( w, m_x, m_y \) and \( m_{xy} \) at each nodal point.

**Mixed Model II.**

Here functional \( R_2 \) is used to develop the element. Again bilinear approximations are employed for each of the three variables. At each of the four corner nodes there exist three (\( w, m_x, \) and \( m_y \)) degrees of freedom (see Figure 1b) resulting in a 12 by 12 element stiffness matrix. The quadratic element contains eight nodes with three degrees of freedom per node.

**Mixed Model III.**

This element is based on functional \( R_2 \). Here bilinear approximations are used for the transverse displacement \( w \), and linear approximations for \( m_x \) and \( m_y \). The four corner nodes each have one displacement degree of freedom. Midnodes on the sides perpendicular to the y-axis each have one degree of freedom \( m_y \), and midnodes on the sides perpendicular to the x-axis each have one degree of freedom \( m_x \) per node (see Figure 1c). This element results in a 8 by 8 element stiffness matrix. Note that \( m_x \) is linear along x but constant along y and \( m_y \) is linear along y and constant along x.

The trial functions (or approximating functions) are shown in Figure 1. For lack of space the element stiffness matrices for each type are not given here. The element matrices are assembled in the usual manner (see Zienkiewicz (8) and Oden and Reddy (9), and the resulting global set of equations are solved for the unknown nodal values.

**NUMERICAL RESULTS**

The above three types of mixed rectangular elements are now used to solve simple problems. Square plates (of length \( L \)) with three types of edge conditions, simply-supported, clamped, and two opposite edges simply-supported and the other two clamped, are solved for uniformly distributed load and concentrated load at the center of the plate. The results are compared with each other and also with those from a hybrid model of Allman (10) and a compatible displacement model of Clough and Tocher (11). Table 1 shows a comparison of center displacements and bending moments for a simply-supported plate with uniform loading. In Table 2 central deflection and bending moments for a clamped square plate under uniform load-
are compared with those for the mixed model IV of Herrmann (1), hybrid model of Allman (10), and conventional cubic displacement model of Clough and Tocher (11). Table 3 contains results for the same plate under concentrated load at the middle. In Table 4 results are presented for a simply-supported plate under concentrated load. Finally Table 5 contains values of central deflection and bending moment, and bending moment at the center of side, for a square plate with two opposite sides clamped and the other two simply-supported. Results are given for uniform loading as well as for concentrated load.

An examination of the results presented indicate that the mixed models I, II, and III described herein are giving better ac-

\[ \psi_1 = (1-\xi)(1-\eta), \, \psi_2 = \xi(1-\eta) \]
\[ \psi_3 = \xi\eta, \, \psi_4 = (1-\xi)\eta \]
\[ \psi_5 = 1-\eta, \, \psi_6 = \xi, \, \psi_7 = \eta, \, \psi_8 = 1-\xi \]

**Figure 1.** Rectangular plate bending elements, finite element mesh of plate, and various mesh types.
### Table 1. Simply-supported square plate under uniformly distributed load, P.

<table>
<thead>
<tr>
<th>Mesh Size</th>
<th>Mixed Model I Linear</th>
<th>Quadratic</th>
<th>Mixed Model II Linear</th>
<th>Quadratic</th>
<th>Mixed Model III Linear</th>
<th>Quadratic</th>
<th>Mixed Model IV Linear</th>
<th>Quadratic</th>
<th>Hybrid Model (10)</th>
<th>Compatible Cubic Displ. Model (11)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1x1</td>
<td>0.4613(16)</td>
<td>0.3906(12)</td>
<td>0.3867(24)</td>
<td>0.4943(8)</td>
<td>0.9018(6)</td>
<td>0.347(12)</td>
<td>0.220(12)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2x2</td>
<td>0.4237(36)</td>
<td>0.4082(27)</td>
<td>0.4053(65)</td>
<td>0.4289(21)</td>
<td>0.512(15)</td>
<td>0.392(27)</td>
<td>0.371(27)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4x4</td>
<td>0.4106(100)</td>
<td>0.4069(75)</td>
<td>0.4062(195)</td>
<td>0.417(65)</td>
<td>0.4516(45)</td>
<td>0.403(75)</td>
<td>0.302(75)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6x6</td>
<td>0.4082(196)</td>
<td>0.4066(147)</td>
<td>0.4062(399)</td>
<td>0.4087(133)</td>
<td>0.4174(101)</td>
<td></td>
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</tr>
</tbody>
</table>

**Bending moment at the center M_x × 10^7/PL^2 (0.479)^a**

<table>
<thead>
<tr>
<th>Mesh Size</th>
<th>Mixed Model I Linear</th>
<th>Quadratic</th>
<th>Mixed Model II Linear</th>
<th>Quadratic</th>
<th>Mixed Model III Linear</th>
<th>Quadratic</th>
<th>Mixed Model IV Linear</th>
<th>Quadratic</th>
<th>Hybrid Model (10)</th>
<th>Compatible Cubic Displ. Model (11)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1x1</td>
<td>0.5193</td>
<td>0.4875</td>
<td>0.2056</td>
<td>0.2487</td>
<td>0.208</td>
<td>0.344</td>
<td></td>
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<tr>
<td>2x2</td>
<td>0.3166</td>
<td>0.2899</td>
<td>0.2248</td>
<td>0.2432</td>
<td>0.242</td>
<td>0.314</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4x4</td>
<td>0.2478</td>
<td>0.2443</td>
<td>0.2287</td>
<td>0.2359</td>
<td>0.235</td>
<td>0.230</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6x6</td>
<td>0.2374</td>
<td>0.2358</td>
<td>0.2290</td>
<td>0.2313</td>
<td>0.232</td>
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</tr>
</tbody>
</table>

**Bending moment at the center M_x × 10^7/PL^2 (0.231)^a**

### Table 2. Clamped square plate under uniformly distributed load, P.

### Table 3. Clamped square plate under concentrated load, P_o.

### Table 4. Simply-supported square plate under concentrated load, P_o.

**a** Exact solution from Reference 13.

**b** Numbers in parenthesis indicate the number of degrees of freedom in the mesh.

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accuracies for the displacement and moments than the mixed model of Herrmann (1), the hybrid model of Allman (10), and the compatible cubic displacement models of Clough and Tocher (11) and Fraeijs de Veubeke and Sander (12). Among the three mixed models described here, mixed model II gives the best accuracies. Although mixed model I gives better accuracies than mixed model III, it requires more storage and computational times. Thus there is a compromise between accuracy and the computational time involved.

**SUMMARY AND CONCLUSIONS**

Three types of rectangular plate bending finite elements are described and compared with each other and also with other mixed, hybrid and compatible finite element models in terms of accuracy and the number of unknowns used in each mesh (which is proportional to the computational time). Square plates with various edge conditions and loadings are analyzed numerically using all three models. It is concluded from the present numerical analysis that the models described herein are economical and give more accurate results. Another advantage which cannot be judged from the numerical results is the very little amount of time needed to compute the element matrices exactly.

Application of these elements to vibration and stability of plates is under way and results will appear elsewhere. Extensions to orthotropic or more general anisotropic plates can be done with very little effort. Use of similar formulation for large deflection analysis of plates is straightforward.

**ACKNOWLEDGMENTS**

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**REFERENCES**


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**Table 5. Clamped simply-supported square plate**

<table>
<thead>
<tr>
<th>Mesh Size</th>
<th>Under uniformly distributed load, P</th>
<th>Under concentrated load P₀</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Central deflection WD x 10⁶/PL¹  (0.192)</td>
<td>Central deflection WD x 10⁶/P₀ L²  (0.7071)</td>
</tr>
<tr>
<td></td>
<td>Mixed Model I</td>
<td>Mixed Model II</td>
</tr>
<tr>
<td>1x1</td>
<td>0.2446 0.2016 0.2034 0.3214 0.9785 0.8065 0.7564 1.2860</td>
<td>0.7341</td>
</tr>
<tr>
<td>2x2</td>
<td>0.2168 0.1963 0.2017 0.2202 0.9805 0.7900 0.7037 0.9041</td>
<td>0.7553 0.7042 0.7652</td>
</tr>
<tr>
<td>4x4</td>
<td>0.1987 0.1937 0.1917 0.1958 0.7279 0.7200 0.7041 0.7341</td>
<td>0.7279 0.7200 0.7041 0.7341</td>
</tr>
<tr>
<td>6x6</td>
<td>0.1963 0.1937 0.1917 0.1958 0.7279 0.7200 0.7041 0.7341</td>
<td>0.7279 0.7200 0.7041 0.7341</td>
</tr>
</tbody>
</table>